Commentary 2: On Random Variability of Responses – A Note on Jens Mammen's Book

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This note relates to only one of Jens Mammen's many themes: the construction of a "new mathematics, tailored for psychology," of which the book shows us only "a small, but important, part" (p. 58). In a nutshell, Mammen proposes to consider an ambient set of "objects" and to endow it with two sets of subsets: the subsets representing (or simply called) "sense categories" and those representing (called) "choice categories." The two sets of subsets are disjoint, except for the empty set that belongs to both. The nonempty sense categories are infinite, and together with the empty set, they form a certain topology. The choice categories may be finite, and a choice category always contains a one-element choice category. The set of choice categories is required to be closed under finite intersections and unions only. The only relationship between the two types of categories is that the intersection of a choice category with a sense category is a choice category.

This is a surprisingly subtle axiomatic construction, with many possible interpretations and directions of specialization (pp. 87–88). A critical evaluation of this construction is beyond the aims of this note. Rather, as a psychophysicist, I am interested in possible operational meanings of this and other similarly abstract constructions. How, by what empirical procedures, does one acquire knowledge of a sense category or a choice category? The book does not provide much detail in this respect. We read, however, that "[i]n mathematical terms decisions about membership of classes or categories could be seen as continuous mappings of a domain of objects on a discrete set of decisions, ultimately on a 'yes/no' set" (Footnote 2 on p. 63). I will assume that I can translate (or generalize) the term "decision" into "response," meaning any form or aspect of observable behavior (including physiological reactions). This will place the discussion in a familiar conceptual framework: whatever the theoretical picture one wants to construct in psychology, one has to ultimately relate it to recordable behavior under recordable conditions. That this time-honored position does no vio-

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lence to Mammen's theory is further evidenced by his saying that the representation of categories by subsets of a set of objects "could also be seen as an 'ecological' generalization of classic experimental psychophysics" (ibid).

The admission that the content (equivalently, meaning) of a category is reflected in, if not determined by, recordable responses given by a person or organism to certain sets of objects has an obvious consequence: if the responses change, the contents (meanings) of the categories generally change too. This leads me to this commentary's departure point: most responses do change, from one instance of presentation to another or from one respondent to another. If one presents many times the same pair of physically very close (or even identical) color patches to a person and asks whether they are the same or different in color, the response will sometimes be yes and sometimes no. If many people are asked whether they trust a certain political party, the response will differ from one person to another. These two types of variability (famously labeled by Thurstone, 1927, as Case I and Case II, respectively) are so ubiquitous that they can be called fundamental. A philosopher may disagree that the latter term can be used in psychology in the same sense in which it is used in relation to quantum mechanics, but this makes little difference: in most psychological situations, the best one can hope to do is to evaluate and, with the help of a theory, predict probabilities of occurrences of various responses, rather than responses themselves.

Why is this stochastic variability worth mentioning? Am I not talking about trivialities of an experiment, "statistical noise" or "errors of measurement"? Is it not something to be carefully isolated from an "ideal" picture, like the one proposed by Mammen, and in no way affecting its essential features? I do not think so. The most conspicuous difference between modern psychophysics and psychophysics predating the 1950s is that in the former the probabilistic aspects of responses are treated as an essential and irreducible part of the experimental paradigms one deals with, such as detection, discrimination, or identification. The same can be said about many other areas of psychology, especially decision making, whether low level or high level. Random variables associated with typical responses used in experiments are simple and well understood, especially categorical ones (those with a finite number of possible values), and among those, especially dichotomous ones, such as yes/no choices. As a result, not only are not they a nuisance factor obscuring an ideal theoretical picture, their probability distributions provide critical information helping one to construct such pictures, being in many cases the only such information available.

Thus, if a person is presented elements of a Mammen's set of objects $\grave{U}$ pairwise, and for each ordered pair (x, y) says whether y is greater than x in a designated respect (such as brightness or beauty), then the probability of the positive response γ transforms the set of objects into a structured space $(\grave{U}, \gamma)$. (Of course, if we speak of a realistic experiment, each pair should be presented many times, and probabilities should be statistically estimated from frequencies and theoretically interpolated/extrapolated to all possible pairs—all well-known difficult issues I am going to gloss over as they are not specific to the present discussion.) Due to the complete transparency of the function γ, one can investigate the properties of this space without first understanding the physical properties of the objects and their perceptual representations. Thus, one can first relabel the objects to get rid of all "true duplicates" (using Mammen's term but in relation to γ only): this means that if

$$\gamma(x_1,y)=\gamma(x_2,y)$$

for all y, then x_1 and x_2 are "lumped" and considered the same object. For y-objects the "lumping" is similar. Following the tradition, one defines a match (point of subjective equality) for x as the object y for which $\gamma(x,y)=1/2$, with a match for y defined symmetrically. One can then check if the space is "regular" (Dzhafarov & Dzhafarov, 2010), which in relation to ($\grave{U}$, γ) means that (after the "lumping") x is matched by one and only one y and vice versa. It is easy to see then that in a regular space x matches y if and only if y matches x, and this allows one to further relabel either the x-objects or the y-objects to bring the space to a canonical form in which $\gamma(x, y)$ equals $1/2$ if and only if $x = y$.

The quantities

$$D(x,y)=|\gamma(x,y)-1/2|$$

are called psychometric increments (from x to y). By observing (or postulating) various properties of D, one can keep imposing on the space of objects progressively more informative (more restrictive) structures. Thus, it is clear that $D(x, x) = 0$ and that $D(x, y) > 0$ if x and y are distinct. We say that sequences $x_1, x_2,\ldots$ and $x'_1, x'_2,\ldots$ converge to each other if $D(x_n,x'_n)$ converges to zero (as $n \to \infty$). One can further assume (provided no contradicting empirical evidence is found) that for any sequences $\{x_n\}$ and $\{x'_n\}$ converging to each other and any sequences $\{y_n\}$ and $\{y'_n\}$ converging to each other:

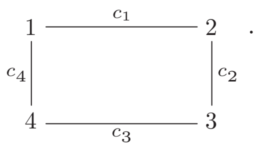
$$D(x_n,y_n)-D(x'_n,y'_n)\to 0.$$

This so-called intrinsic uniform convergence property allows one to impose on $\grave{U}$ a structure called uniformity, which in turn induces a certain form of topology.

In a series of publications (see Dzhafarov, 2011, for an overview), it has been shown how, with the addition of one other property whose description I omit as it is more technical, the notion of the dissimilarity function D can be used to impose on $\grave{U}$ a metric structure and develop a full-fledged geometry. The operational part of the development in most of these publications is different from the one presented here: the dissimilarity function there is based on mathematically more complex but operationally more versatile "same-different" judgments rather than "greater-less" judgments (Dzhafarov & Colonius, 1999, 2007). One can also define D through other procedures, such as adjustment of y until it seems matching x (Dzhafarov & Perry, 2014). All of these procedures deal with randomly varying responses. We need not go further into details to appreciate the fact that this random variability, far from being a nuisance factor, is part and parcel of the construction of the "ideal" pictures of the kind presented in Mammen's book.

My other illustration of the necessity to treat random variability as essential part of one's theoretical construction relates to the issue of (probabilistic) contextuality. Mammen touches on this issue very briefly but prominently on pp. 23–24, when discussing quantum entanglement as having dealt a "deathblow" to the idea of the universe being driven by local mechanical interactions. This particular paradigm of contextuality is schematically shown below.

Two particles (say, electrons), created and "entangled" within a certain state called "singular," move away from each other, and at some moment of time, Anna and Curt measure their spins, Anna in the left electron and Curt in the right one. Spin is a property of an elementary particle (here, electron) tied to a choice of an axis (direction) in space: thus, for the left particle, one can speak of its spin in direction 1 or in direction 3, and Anna always chooses one of these two axes. Similarly, Curt always chooses one of the two axes labeled 2 and 4 for the right particle. The labeling of the axes in the two particles is chosen so that the possible pairs of the axes simultaneously chosen by Anna and Curt form a cycle.



Each such a pair of the axes is called a context. So in the paradigm considered, we have four axes combined into four contexts, c_1, c_2, c_3, and c_4.

In electrons, the outcomes of measuring a spin along any axis is binary, and one can denote these outcomes arbitrarily, say, "yes" and "no." Let us denote by R_q^c the yes/no random variable representing the result of measuring spin along axis q in context c. We have therefore eight random variables, as shown in the matrix below:

$$\begin{bmatrix} (\text{EPR B}) & c_1 & c_2 & c_3 & c_4 \\ 1 & R_1^1 & & & R_1^4 \\ 2 & R_2^1 & R_2^2 & & \\ 3 & & R_3^2 & R_3^3 & \\ 4 & & & R_4^3 & R_4^4 \end{bmatrix}.$$

The abbreviation EPR stands for Einstein, Podolsky, and Rosen (1935), and B stand either for Bohm, who adapted the EPR problem to spins (Bohm & Aharonov, 1957), or Bell (1964, 1966), who famously investigated it. The question that interested Bell and two generations of quantum physicists after him, when formulated in the language of the representation above, is this (Dzhafarov & Kujala, 2014, 2017b): can one present all the random variables in the system as functions of one and the same ("hidden") random variable, so that any two random variables measuring spin for the same axis in two different context are the same? In other words, can one find a random variable R such that every R_q^c in the system can be presented as some function f_q (R) (without the superscript c)? As it turns out, the axes 1, 2, 3, and 4 can be chosen so that the answer to this question is negative. For instance, in accordance with the laws of quantum mechanics, coplanar axes 1, 2, 3, and 4 forming the respective angles 0, $\pi/4$, $\pi/2$, and $-\pi/4$ with a horizontal line yield

$$\begin{bmatrix} context & & & \\ c_1=(1,2) & & & \\ & R_2^1=Yes & R_2^1=No & \\ R_1^1=Yes & (2+\sqrt{2})/8 & (2-\sqrt{2})/8 & 1/2 \\ R_1^1=No & (2-\sqrt{2})/8 & (2+\sqrt{2})/8 & 1/2 \\ & 1/2 & 1/2 & \end{bmatrix} \begin{bmatrix} context & & & \\ c_4=(4,1) & & & \\ & R_4^4=Yes & R_4^4=No & \\ R_1^4=Yes & (2-\sqrt{2})/8 & (2+\sqrt{2})/8 & 1/2 \\ R_1^4=No & (2+\sqrt{2})/8 & (2-\sqrt{2})/8 & 1/2 \\ & 1/2 & 1/2 & \end{bmatrix}$$

$$\begin{bmatrix} context & & & \\ c_2=(2,3) & & & \\ & R_2^2=Yes & R_2^2=No & \\ R_3^2=Yes & (2+\sqrt{2})/8 & (2-\sqrt{2})/8 & 1/2 \\ R_3^2=No & (2-\sqrt{2})/8 & (2+\sqrt{2})/8 & 1/2 \\ & 1/2 & 1/2 & \end{bmatrix} \begin{bmatrix} context & & & \\ c_3=(3,4) & & & \\ & R_4^3=Yes & R_4^3=No & \\ R_3^3=Yes & (2+\sqrt{2})/8 & (2-\sqrt{2})/8 & 1/2 \\ R_3^3=No & (2-\sqrt{2})/8 & (2+\sqrt{2})/8 & 1/2 \\ & 1/2 & 1/2 & \end{bmatrix}$$

Here, the first matrix corresponds to context c_1, in which Anna chooses axis 1 and Curt chooses axis 2. The cell for $R_1^1=Yes$ and $R_2^1=No$ shows the joint probability of these two outcomes, and the cell for $R_1^1=Yes$ on the margin shows the "marginal" probability of this outcome, irrespective of the other measurement. Other cells in this and other contexts are interpreted analogously. It can be shown that even though every pair R_q^c and $R_q^{c'}$ measuring spin along the same axis in different contexts are identically distributed (i.e., in the parlance of quantum mechanics, the system is "non-signaling"), there is no way of presenting all eight random variables as functions of a single background variable R that would make R_q^c and $R_q^{c'}$ indistinguishable (i.e., always equal to each other). At least for some axes q, the variables R_q^c and $R_q^{c'}$ must be presented as different functions $f_q^c(R)$ and $f_q^{c'}(R)$. One can say that the two random variables have different identity due to different contexts, and this difference cannot be ignored even though it does not translate into different distributions. We say in cases like this that the system of random variables is contextual, in the traditional quantum-mechanical understanding of the term.

Contextuality is a deep concept that cannot be confined to nonlocality and entanglement only. For example, the KCBS (Klyachko, Can, Binicioglu, & Shumovsky, 2008) system with five axes pairwise used in five contexts has essentially the same mathematical structure:

$$\begin{bmatrix} (\mathrm{KCBS}) & c_1 & c_2 & c_3 & c_4 & c_5 \\ 1 & R_1^1 & & & & R_1^5 \\ 2 & R_2^1 & R_2^2 & & & \\ 3 & & R_3^2 & R_3^3 & & \\ 4 & & & R_4^3 & R_4^4 & \\ 5 & & & & R_5^4 & R_5^5 \end{bmatrix},$$

and lends itself to essentially the same contextuality analysis, even though all the measurements are performed on a single particle. The same applies to the SZ-LG (Suppes & Zanotti, 1981; Leggett & Garg, 1985) system with a single particle measured at three fixed moments of time grouped pairwise into three contexts:

$$\begin{bmatrix} (\mathrm{SZ\ \ LG}) & c_1 & c_2 & c_3 \\ q_1 & R_1^1 & & R_1^3 \\ q_2 & R_2^1 & R_2^2 & \\ q_3 & & R_3^2 & R_3^3 \end{bmatrix}.$$

Returning to the EPR-B paradigm, it is clear that one can create a system of the same formal structure outside quantum mechanics. The axes 1, 2, 3, and 4 can be replaced, e.g., with yes/no questions asked of a person or many people. Mammen has written a delightful essay (Mammen, 2016) in which Anna and Curt instead of measuring spins answer questions asked of them simultaneously in two different Danish cities (which is the reason I use these names instead of the traditional "Alice" and "Bob"). The scenario Mammen considers is as follows:

$$\begin{bmatrix} context & & & \\ c_1 = (1,2) & & & \\ & R_2^1 = Yes & R_2^1 = No & \\ R_1^1 = Yes & 1/2 & 0 & 1/2 \\ R_1^1 = No & 0 & 1/2 & 1/2 \\ & 1/2 & 1/2 & \end{bmatrix} \quad \begin{bmatrix} context & & & \\ c_4 = (4,1) & & & \\ & R_4^4 = Yes & R_4^4 = No & \\ R_1^4 = Yes & 0 & 1/2 & 1/2 \\ R_1^4 = No & 1/2 & 0 & 1/2 \\ & 1/2 & 1/2 & \end{bmatrix}$$

$$\begin{bmatrix} context & & & \\ c_2 = (3,2) & & & \\ & R_2^2 = Yes & R_2^2 = No & \\ R_3^2 = Yes & 1/2 & 0 & 1/2 \\ R_3^2 = No & 0 & 1/2 & 1/2 \\ & 1/2 & 1/2 & \end{bmatrix} \quad \begin{bmatrix} context & & & \\ c_3 = (3,4) & & & \\ & R_4^3 = Yes & R_4^3 = No & \\ R_3^3 = Yes & 1/2 & 0 & 1/2 \\ R_3^3 = No & 0 & 1/2 & 1/2 \\ & 1/2 & 1/2 & \end{bmatrix}$$

In the quantum-mechanical literature, this system is referred to as a PR box (after Popescu & Rohrlich, 1994). It is a system with maximal algebraically possible contextuality, and it cannot be realized by any quantum-mechanical system. Of course, nor can this or any other contextual system be realized by the humorous situation described in Mammen's essay. However, there seems to be no a priori reasons why a contextual system cannot be realized if the paired questions are asked of the same person. The point that is important in the present discussion is that any chance of finding a contextual system in biological or social behavior is contingent on the system being stochastic: a deterministic system is always noncontextual.

One safe generalization about biological and social behavior is that a response to object x is always directly (causally) influenced by any other object in spatial-temporal proximity of x. If a person answers question 1, the distribution of her responses will be different depending on whether together with 1 she is asked question 2 or question 4. This has been demonstrated in numerous attempts to recreate the formal structure of EPR-B, SZ-LG, and other similar systems in human behavior (see Dzhafarov, Zhang, & Kujala, 2015, for an overview). This means that unlike in the traditional quantum-mechanical definition, we cannot stipulate "non-signaling" as a necessary condition for contextuality analysis, unless we want the result of this analysis to be trivially predetermined. Fortunately, there are compelling reasons for and natural ways of generalizing the definition of (non)contextual systems to include those with "signaling" (Dzhafarov & Kujala, 2015, 2017a, 2017b; Kujala, Dzhafarov, Larsson, 2015). In regard to the EPR-B system, the generalized definition would be: The system is noncontextual if each of the eight random variables R_q^c in it can be presented as functions of one and the same "hidden" variable R, so that $f_q^c(R)$ and $f_q^{c'}(R)$ coincide with the maximal possible probability (this maximal probability being 1 if and only if R_q^c and $R_q^{c'}$ are identically distributed). The statement that is relevant to this note, however, remains valid: a deterministic system cannot be contextual, whether "signaling" or not. Therefore any procedure, such as averaging, that would eliminate variability as a nuisance factor would also eliminate contextuality.

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